Laplace Transform

Definition, Standard transforms

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Laplace Transform

- Laplace transformation is a technique for solving differential equations.
- Differential equation of time domain form is first transformed to algebraic equation of frequency domain form.
- After solving the algebraic equation in frequency domain, the result then is finally transformed to time domain form to achieve the ultimate solution of the differential equation.
- In other words it can be said that the Laplace transformation is nothing but a shortcut method of solving differential equation.

Laplace Transform Definition

To understand the Laplace transform formula: First Let f(t) be the function of t, time for all t ≥ 0

Then the Laplace transform of f(t), F(s) can be defined as

$$\mathbf{L}[f(t)] = F(s) = \int_0^{\infty} f(t)e^{-st}\,dt$$

Provided that the integral exists. Where the Laplace Operator, s = σ + jω; will be real or complex j = √(-1)

Laplace Transform of Some Basic Functions

Unit Step Signal $f(t) = 1 \text{ (for } t \geq 0)$

$$F(s) = \int_0^\infty e^{-st}\, dt$$

$$= -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s}$$

impulse function, $f = \delta$

if *f* contains impulses at t = 0 we choose to include them in the integral defining F:

$$F(s) = \int_{0-}^\infty f(t)e^{-st}\, dt$$

$$F(s) = \int_{0-}^{\infty} \delta(t) e^{-st}\, dt \quad = e^{-st}\Big|_{t=0} = 1$$

similarly for $f = \delta^{(k)}$ we have

$$F(s) = \int_{0-}^{\infty} \delta^{(k)}(t) e^{-st}\, dt$$

$$= (-1)^k \frac{d^k}{dt^k} e^{-st}\Big|_{t=0}$$

$$= s^k e^{-st}\Big|_{t=0} = s^k$$

$$f(t) = e^t$$

$$F(s) = \int_0^\infty e^t \, e^{-st} \, dt$$

$$= \int_0^\infty e^{(1-s)t} \, dt$$

$$= \frac{1}{1-s} e^{(1-s)t} \Big|_0^\infty$$

$$= \frac{1}{s-1}$$

$f(t) = \cos\omega t$ as

$$f(t) = (1/2)e^{j\omega t} + (1/2)e^{-j\omega t}$$

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st}\left((1/2)e^{j\omega t} + (1/2)e^{-j\omega t}\right)\, dt \\ &= (1/2)\int_0^\infty e^{(-s+j\omega)t}\, dt + (1/2)\int_0^\infty e^{(-s-j\omega)t}\, dt \\ &= (1/2)\frac{1}{s-j\omega} + (1/2)\frac{1}{s+j\omega} \\ &= \frac{s}{s^2+\omega^2} \end{aligned}$$

$$f(t) = t^n \ (n \geq 1)$$

$$F(s) = \int_0^\infty t^n e^{-st} \, dt$$

$$= \ t^n \left(\frac{-e^{-st}}{s} \right) \Big|_0^\infty + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} \, dt$$

$$= \ \frac{n}{s} \, \mathcal{L}(t^{n-1})$$

provided $t^n e^{-st} \to 0$ if $t \to \infty$, which is true for $\Re s > 0$

$$F(s) = \frac{n!}{s^{n+1}}$$

Summary of Laplace Transform

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$t^n, \quad n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$
$\sin(at)$	$\frac{a}{s^2 + a^2}$
$\cos(at)$	$\frac{s}{s^2 + a^2}$

Thank you